EE 505

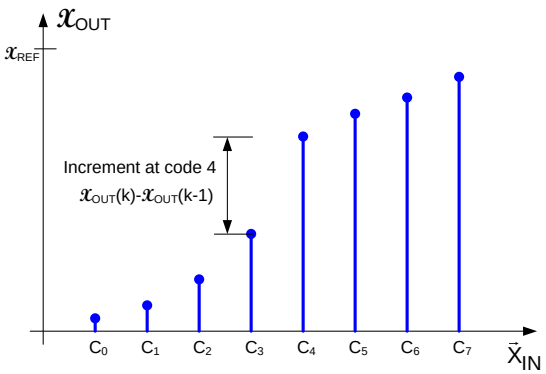
Lecture 4

Quantization Noise
Spectral Characterization

CORRECTION from Last Lecture

Differential Nonlinearity (DAC)

Nonideal DAC



Theorem: The INL_k of a DAC (when corrected for gain error and offset) can be obtained from the DNL by the expression

$$INL_k = \sum_{i=1}^{k} DNL(i)$$

Caution: Be careful about using this theorem to measure the INL since errors in DNL measurement (or simulation) can accumulate

Corollary: The DNL of a DAC (when corrected for gain error and offset) can be expressed as

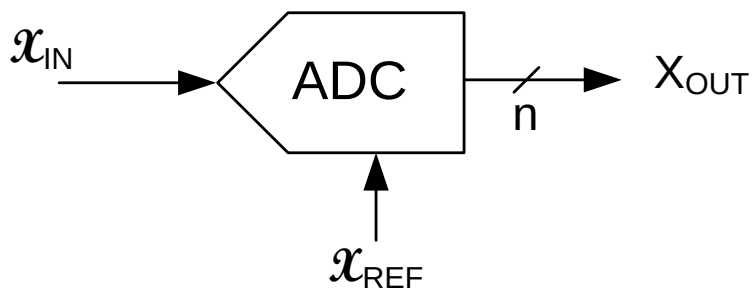
$$DNL(k) = INL_k - INL_{k-1}$$

REVIEW from Last Lecture

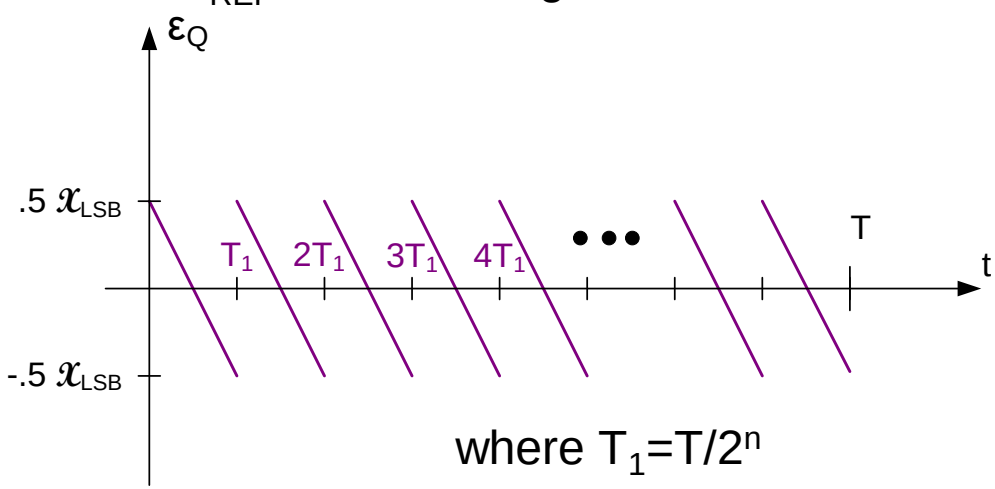
Quantization Noise in ADC

(same concepts apply to DACs)

Consider an Ideal ADC with first transition point at $0.5X_{LSB}$



If the input is a low frequency sawtooth waveform of period T that goes from 0 to X_{REF} , the error signal in the time domain will be:



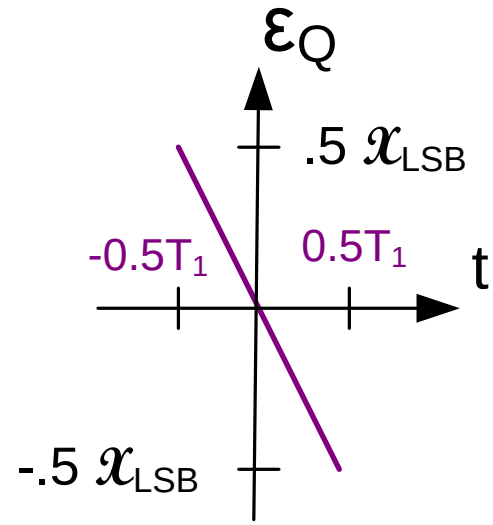
where $T_1=T/2^n$

This time-domain waveform is termed the Quantization Noise for the ADC with a sawtooth (or triangular) input

REVIEW from Last Lecture

Quantization Noise in ADC

$$E_{RMS} = \sqrt{\frac{1}{T_1}\int_{-T_1/2}^{T_1/2} \varepsilon_Q^{\,2}(t)\,dt}$$



$$\varepsilon_Q(t) = -\left(\frac{X_{LSB}}{T_1}\right)t$$

$$E_{RMS} = \sqrt{\frac{1}{T_1}\int_{-T_1/2}^{T_1/2} \left(-\frac{x_{LSB}}{T_1}\right)^2 t^2\,dt}$$

$$E_{RMS} = x_{LSB}\sqrt{\frac{1}{T_1^3}\left.\frac{t^3}{3}\right|_{-T_1/2}^{T_1/2}}$$

$$E_{RMS} = \frac{x_{LSB}}{\sqrt{12}}$$

REVIEW from Last Lecture

Quantization Noise in ADC

$$E_{RMS} = \frac{x_{LSB}}{\sqrt{12}}$$

The signal to quantization noise ratio (SNR) can now be determined. Since the input signal is a sawtooth waveform of period T and amplitude X_{REF}, it follows by the same analysis that it has an RMS value of

$$x_{RMS} = \frac{x_{REF}}{\sqrt{12}}$$

Thus the SNR is given by

$$SNR = \frac{x_{RMS}}{E_{RMS}} = \frac{x_{RMS}}{x_{LSB}} = 2^n$$

or, in dB,

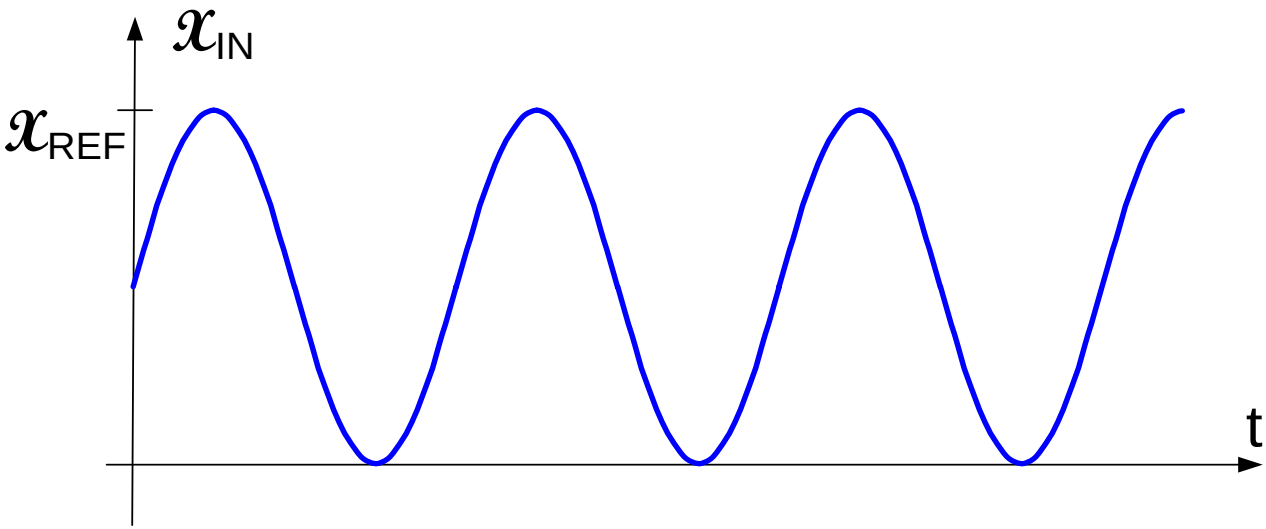
$$SNR_{dB} = 20(n \bullet \log 2) = 6.02n$$

Note: dB subscript often neglected when not concerned about confusion

REVIEW from Last Lecture

Quantization Noise in ADC

How does the SNR change if the input is a sinusoid that goes from 0 to x_{REF} centered at $x_{REF}/2$?

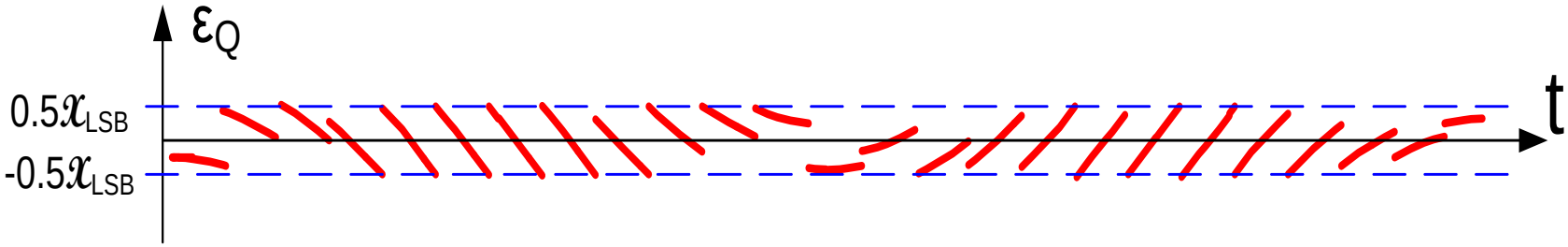


$$SNR = 20(n \bullet \log 2) = 6.02n$$

REVIEW from Last Lecture

Quantization Noise in ADC

How does the SNR change if the input is a sinusoid that goes from 0 to x_{REF} centered at $x_{REF}/2$?



For low f_{SIG}/f_{CL} ratios, bounded by ±XLB and at any point in time, behaves almost as if a uniformly distributed random variable

$$\varepsilon_Q \sim U[-0.5X_{LSB}, 0.5X_{LSB}]$$

REVIEW from Last Lecture

Quantization Noise in ADC

Recall:

If the random variable f is uniformly distributed in the interval [A,B]
f : U[A,B] then the mean and standard deviation of f are given by

$$\mu_f = \frac{A+B}{2} \qquad \sigma_f = \frac{B-A}{\sqrt{12}}$$

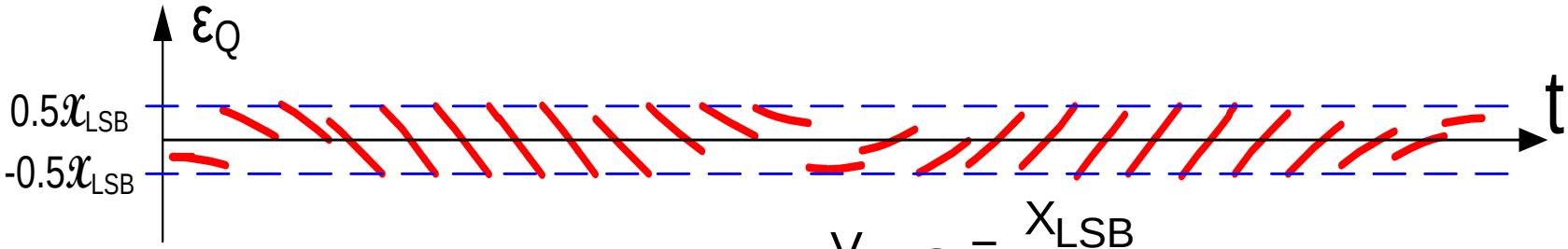
Theorem: If n(t) is a random process, then for large T,

$$V_{RMS} = \sqrt{\frac{1}{T}\int_{t_1}^{t_1+T} n^2(t)\,dt} = \sqrt{\sigma_n^2+\mu_n^2}$$

REVIEW from Last Lecture

Quantization Noise in ADC

How does the SNR change if the input is a sinusoid that goes from 0 to x_{REF} centered at $x_{REF}/2$?



$$V_{RMS} = \frac{X_{LSB}}{\sqrt{12}}$$

But $$V_{INRMS} = \left(\frac{X_{REF}}{2}\right)\frac{1}{\sqrt{2}}$$

Thus obtain

$$SNR = \frac{\dfrac{X_{REF}}{2\sqrt{2}}}{\dfrac{X_{LSB}}{\sqrt{12}}} = 2^n\sqrt{\frac{3}{2}}$$

Finally, in db,

$$SNR_{dB} = 20\log\left(2^n\sqrt{\frac{3}{2}}\right) = 6.02\,n + 1.76$$

REVIEW from Last Lecture

ENOB based upon Quantization Noise

$$SNR = 6.02\ n + 1.76$$

Solving for n, obtain

$$ENOB = \frac{SNR_{dB} - 1.76}{6.02}$$

Note: could have used the SNR_{dB} for a triangle input and would have obtained the expression

$$ENOB = \frac{SNR_{dB}}{6.02}$$

But the earlier expression is more widely used when specifying the ENOB based upon the noise level present in a data converter

REVIEW from Last Lecture

ENOB based upon Quantization Noise

For very low resolution levels, the assumption that the quantization noise is uncorrelated with the signal is not valid and the ENOB expression will cause a modest error

from van de Plassche (p13)

$$SNR_{corr} \cong \left(2^n - 2 + \frac{4}{\pi}\right)\sqrt{\frac{3}{2}}$$

Res (n)	SNR_{corr}	SNR
1	3.86	7.78
2	12.06	13.8
3	19.0	19.82
4	25.44	25.84
5	31.66	31.86
6	37.79	37.88
8	49.90	49.92
10	61.95	61.96

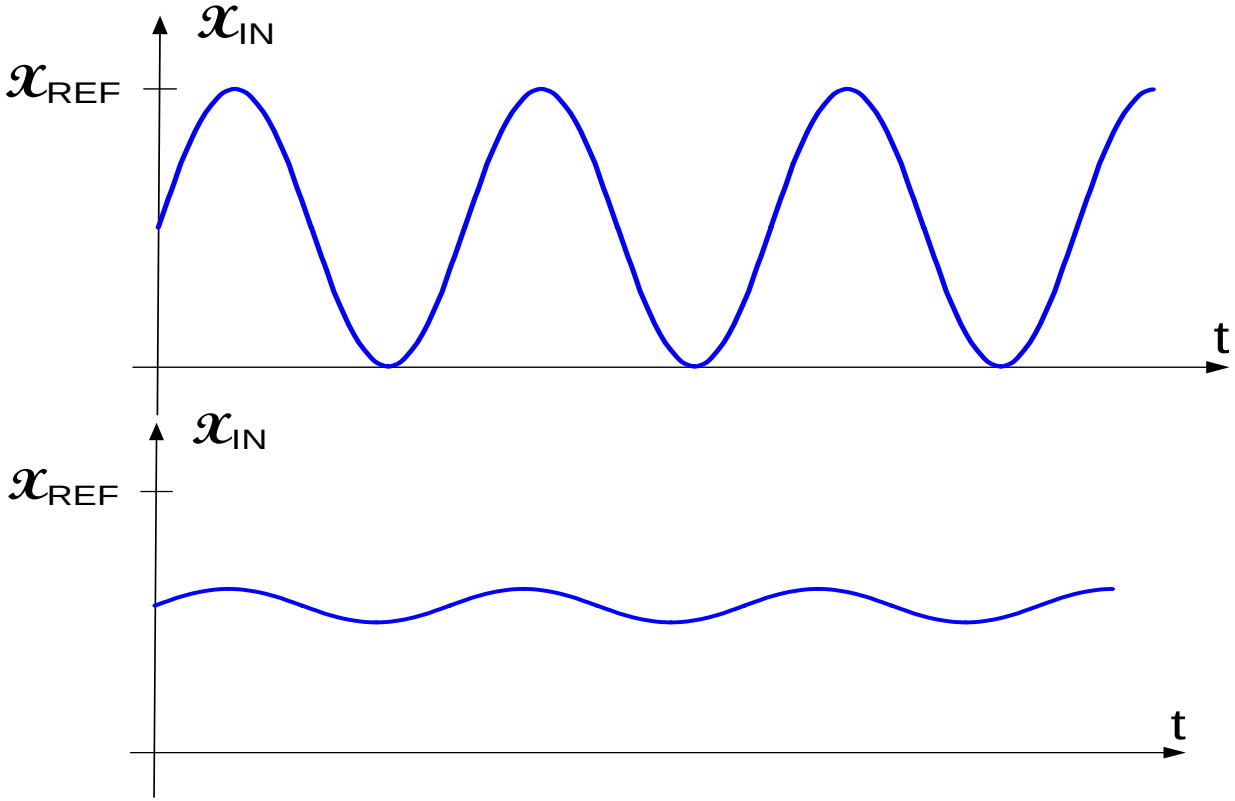
SNR = 6.02 n +1.76

Table values in dB

Almost no difference for n ≥ 3

Quantization Noise

Effects of quantization noise can be very significant, even at high resolution, when signals are not of maximum magnitude

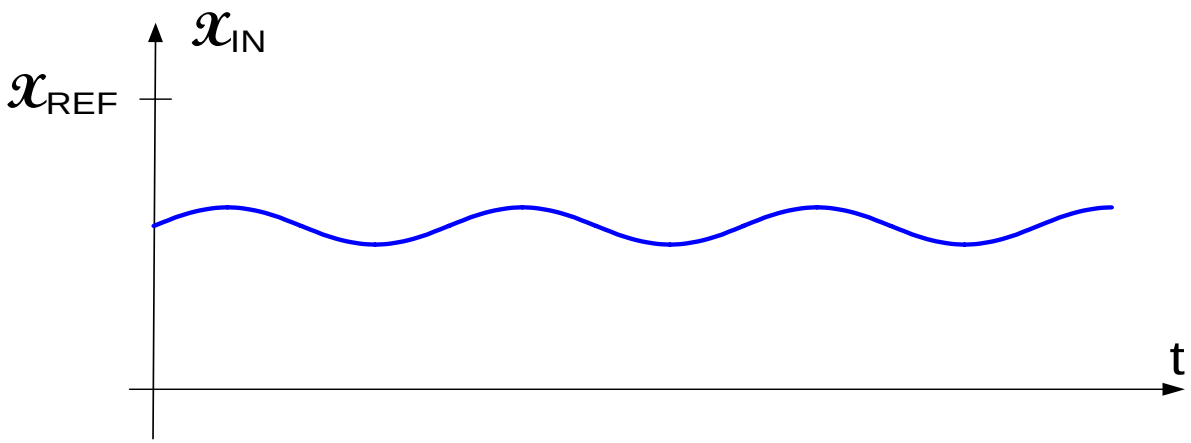


Quantization noise remains constant but signal level is reduced

The desire to use a data converter at a small fraction of full range is one of the major reasons high resolution is required in many applications

Quantization Noise

Effects of quantization noise can be very significant, even at high resolution, when signals are not of maximum magnitude



Quantization Noise

Example: If a 14-bit audio output is derived from a DAC designed for providing an output of 100W but the normal listening level is at 50mW, what is the SNR due to quantization noise at maximum output and at the normal listening level? What is the ENOB of the audio system when operating at 50mW?

At 100W output, SNR=6.02n+1.76 = 90.6dB

$$\frac{V^2}{R_L}=100W \qquad \frac{V_1^2}{R_L}=50mW \qquad V_1=\frac{V}{44.7}$$

$$20\log_{10}V_1=20\log_{10}V-20\log_{10}44.7=20\log_{10}V\ -33dB$$

At 50mW output, SNR reduced by 33dB to 57.6dB

$$ENOB = \frac{SNR_{dB}-1.76}{6.02} = \frac{57.6-1.76}{6.02} = -9.3$$

Note the dramatic reduction in the effective resolution of the DAC when operated at only a small fraction of full-scale.

ENOB Summary

Resolution:

$$\text{ENOB} = \frac{\log_{10} N_{ACT}}{\log_{10} 2} = \log_2 N_{ACT}$$

INL:

$$\text{ENOB} = n_R - \log_2(\upsilon) - 1$$

n_R specified res, v INL in LSB

$$\text{ENOB} = -\log_2(\text{INL}_{REF}) - 1$$

INL_{REF} INL rel to X_{REF}

DNL:

HW problem

Quantization noise:

rel to triangle/sawtooth

$$\text{ENOB} = \frac{\text{SNR}_{dB}}{6.02}$$

$$\text{ENOB} = \frac{\text{SNR}_{dB} - 1.76}{6.02}$$

rel to sinusoid

Most widely used for static characteristics

Additional ENOB will be introduced when discussing dynamic characteristics

Performance Characterization of Data Converters

- Static characteristics
  - Resolution
  - Least Significant Bit (LSB)
  - Offset and Gain Errors
  - Absolute Accuracy
  - Relative Accuracy
  - Integral Nonlinearity (INL)
  - Differential Nonlinearity (DNL)
  - Monotonicity (DAC)
  - Missing Codes (ADC)
  - Quantization Noise
  - Low-f Spurious Free Dynamic Range (SFDR)
  - Low-f Total Harmonic Distortion (THD)
  - Effective Number of Bits (ENOB)
  - Power Dissipation

Absolute Accuracy

Absolute Accuracy is the difference between the actual output and the ideal or desired output of a data converter

The ideal or desired output is in reference to an absolute standard (often maintained by the National Institute of Standards and Technology – NIST) (renamed from National Bureau of Standards in 1988) and could be volts, amps, time, weight, distance, or one of a large number of other physical quantities)

Absolute accuracy provides no tolerance to offset errors, gain errors, nonlinearity errors, quantization errors, frequency rolloff, or noise

In many applications, absolute accuracy is not of a major concern

Absolute accuracy generally dominated by the nonidealities of the reference (a data converter is a ratio-metric device so no fundamental limit on ratio portion)

but … scales, meters, etc. may be more concerned about absolute accuracy than any other parameter

Relative Accuracy

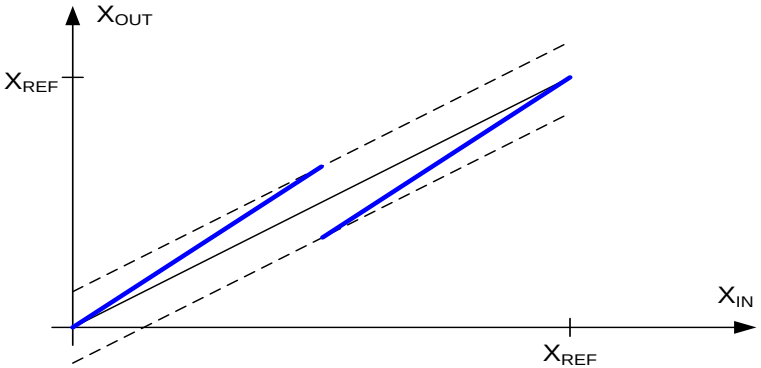
In the context of data converters, pseudo-static Relative Accuracy is the difference between the actual output and an appropriate fit-line to overall output of the data converter

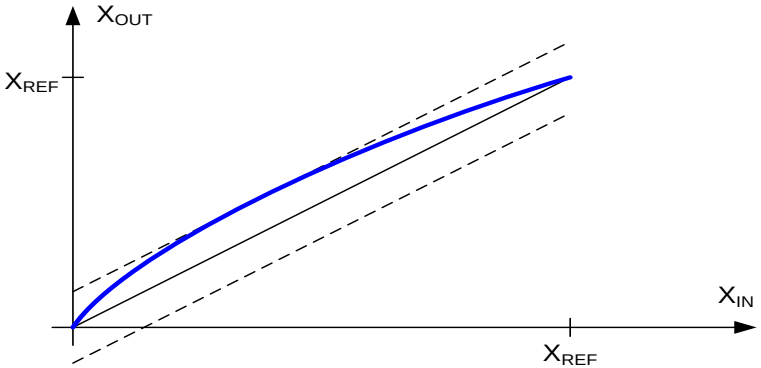
INL is often used as a measure of the relative accuracy

In many, if not most, applications, relative accuracy is of much more concern than absolute accuracy

Some architectures with good relative accuracy will have very small deviations in the outputs for closely-spaced inputs whereas others may have relatively large deviations in outputs for closely-spaced inputs

DNL provides some measure of how outputs for closely-spaced inputs compare





Performance Characterization of Data Converters

- Static characteristics
  - Resolution
  - Least Significant Bit (LSB)
  - Offset and Gain Errors
  - Absolute Accuracy
  - Relative Accuracy
  - Integral Nonlinearity (INL)
  - Differential Nonlinearity (DNL)
  - Monotonicity (DAC)
  - Missing Codes (ADC)
  - Quantization Noise
  - Low-f Spurious Free Dynamic Range (SFDR)
  - Low-f Total Harmonic Distortion (THD)
  - Effective Number of Bits (ENOB)
  - Power Dissipation

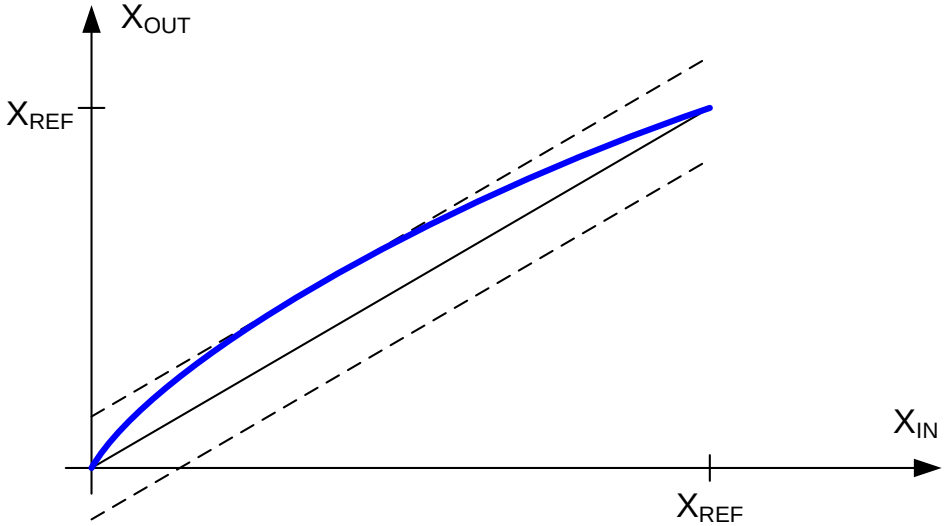
Limitations of INL & DNL in Characterizing Linearity

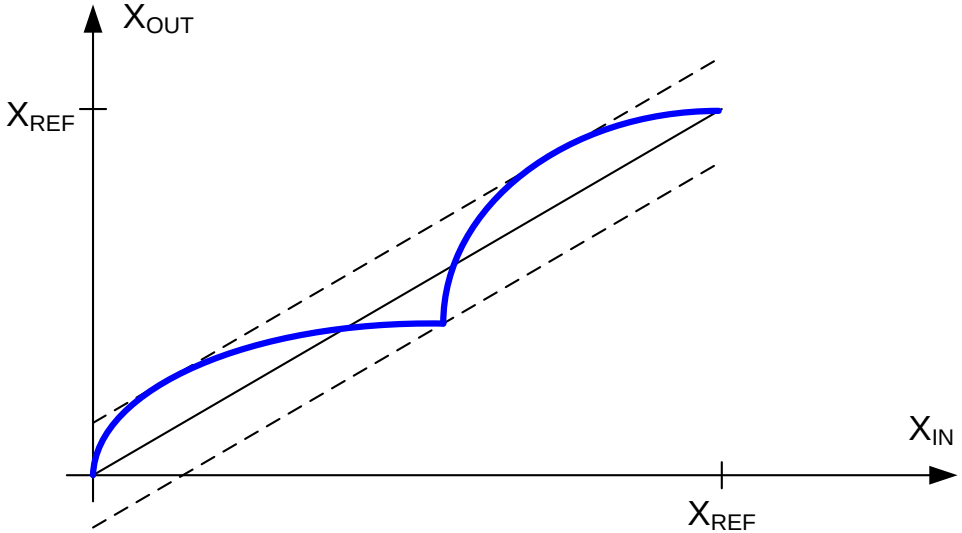
- **INL is a key parameter that is attempting to characterize the overall linearity of a DAC !**
- **INL is a key parameter that is attempting to characterize the overall linearity of an ADC !**
- **DNL is a key parameter that is attempts to characterize the local linearity of a DAC !**
- **DNL is a key parameter that is attempts to characterize the local linearity of an ADC !**

Are INL and DNL effective at characterizing the linearity of a data converter?

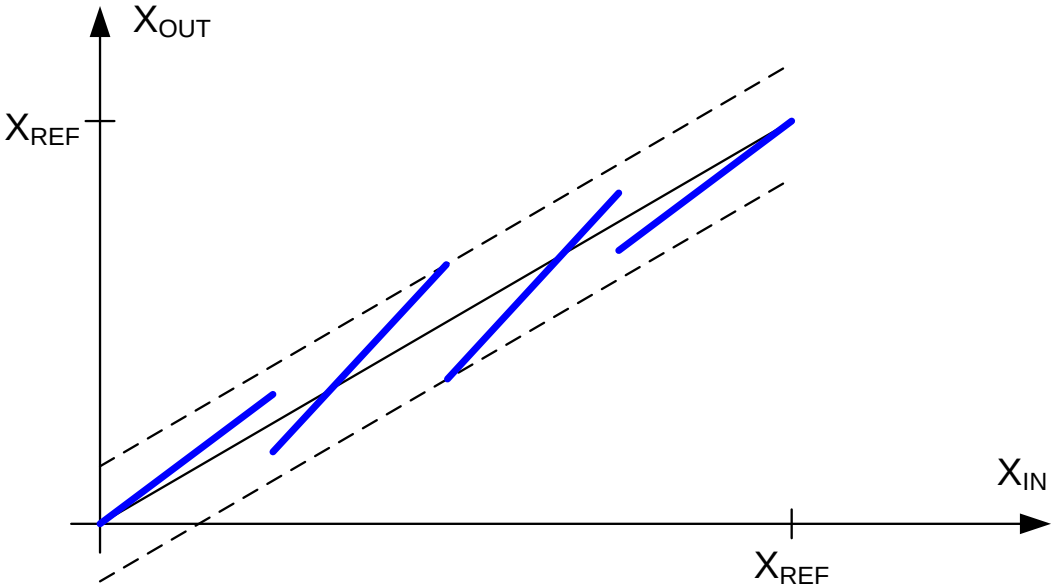
Limitations of INL & DNL in Characterizing Linearity

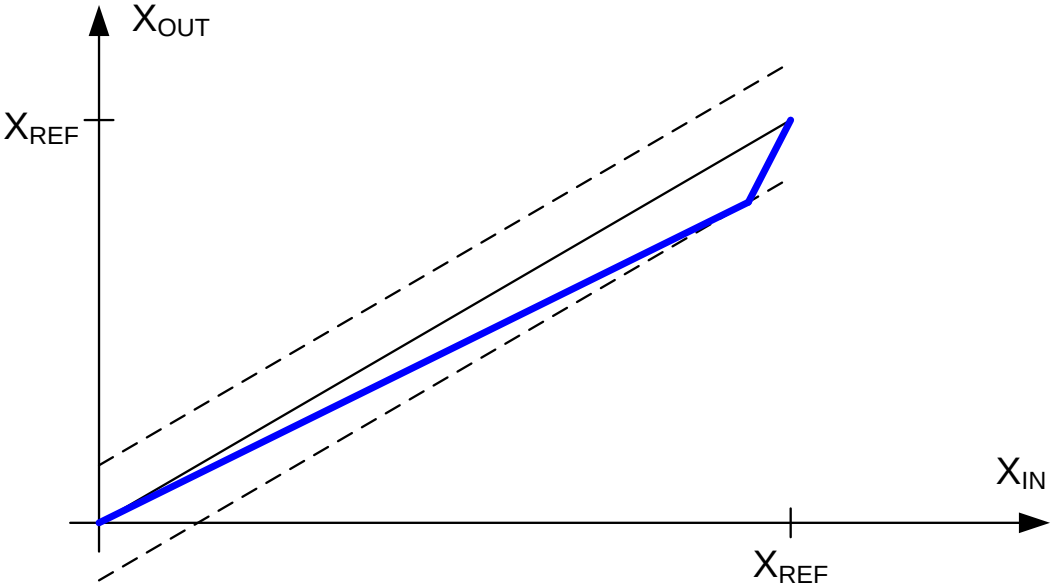
Consider the following 4 transfer characteristics, all of which have the same INL



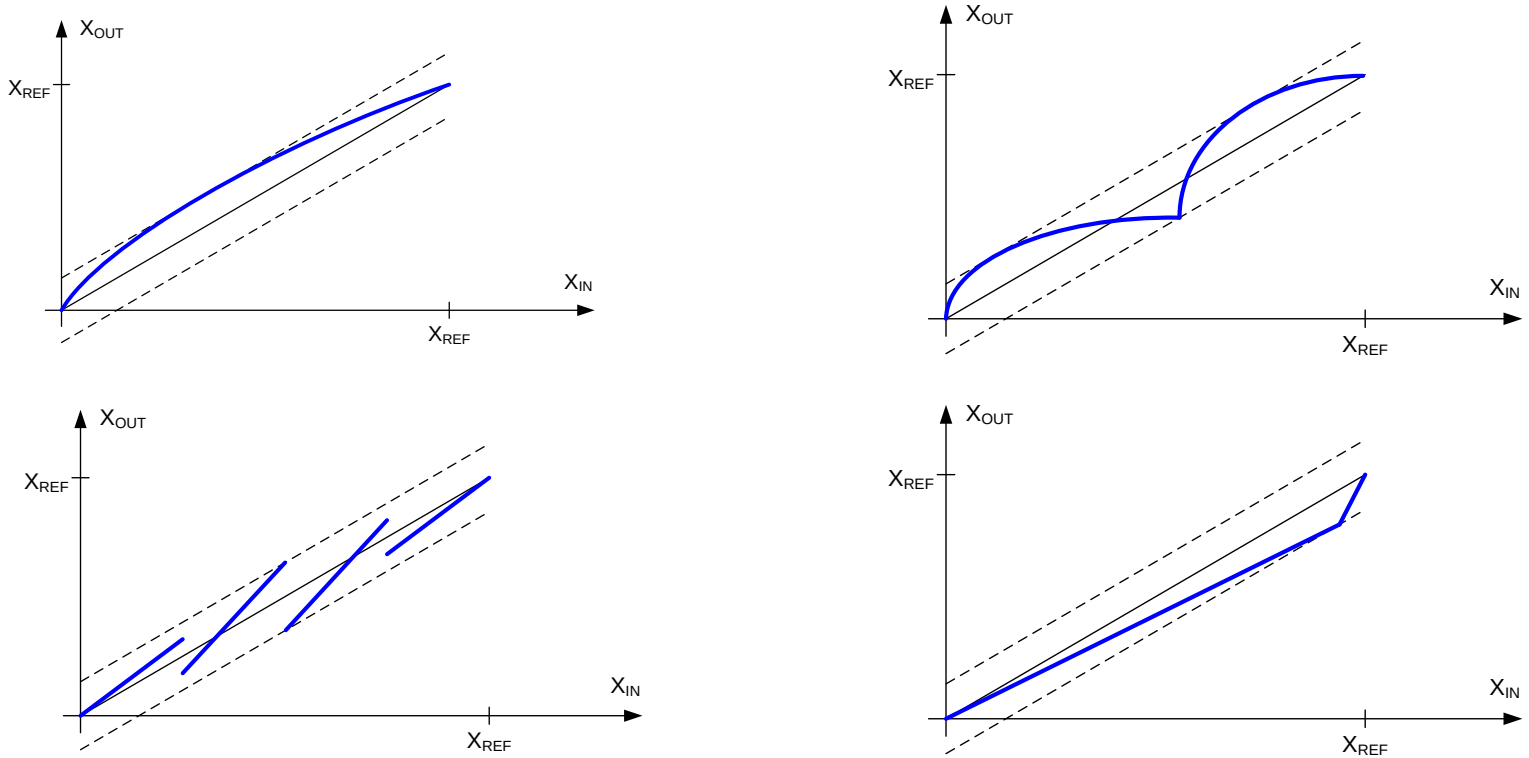


Limitations of INL & DNL in Characterizing Linearity





Limitations of INL & DNL in Characterizing Linearity



Although same INL, dramatic difference in performance particularly when inputs are sinusoidal-type excitations

INL also gives little indication of how performance degrades at higher frequencies

Spectral Analysis often used as an alternative (and often more useful in many applications) linearity measure for data converters

Linearity Issues

- INL is often not adequate for predicting the linearity performance of a data converter
- Distortion (or lack thereof) is of major concern in many applications
- Distortion is generally characterized in terms of the harmonics that may appear in a waveform

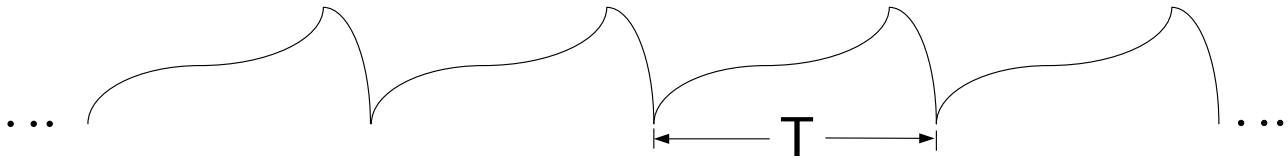
Spectral Analysis often used as an alternative (and often more useful in many applications) linearity measure for data converters

Performance Characterization of Data Converters

- Static characteristics
  - Resolution
  - Least Significant Bit (LSB)
  - Offset and Gain Errors
  - Absolute Accuracy
  - Relative Accuracy
  - Integral Nonlinearity (INL)
  - Differential Nonlinearity (DNL)
  - Monotonicity (DAC)
  - Missing Codes (ADC)
  - Quantization Noise
  - Low-f Spurious Free Dynamic Range (SFDR)
  - Low-f Total Harmonic Distortion (THD)
  - Effective Number of Bits (ENOB)
  - Power Dissipation

Spectral Characterization

Spectral Analysis



If f(t) is periodic

$$f(t) = A_0 + \sum_{k=1}^{\infty} A_k \sin\left(k\omega t + \theta_k\right)$$

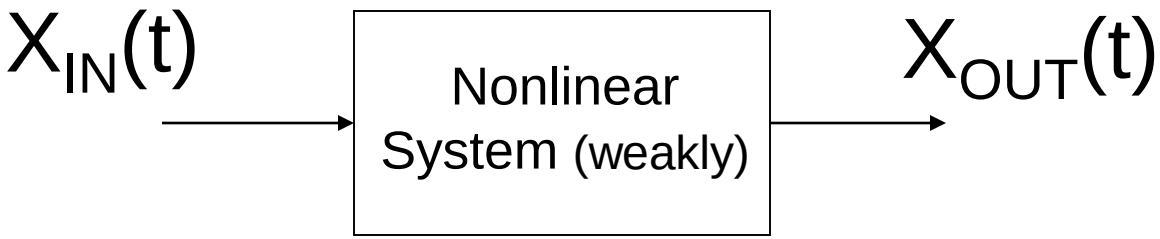
alternately

$$f(t) = A_0 + \sum_{k=1}^{\infty} a_k \sin\left(k\omega t\right) + \sum_{k=1}^{\infty} b_k \cos\left(k\omega t\right) \qquad \omega = \frac{2\pi}{T}$$

$$A_k = \sqrt{a_k^2 + b_k^2}$$

Termed the Fourier Series Representation of f(t)

Spectral Analysis



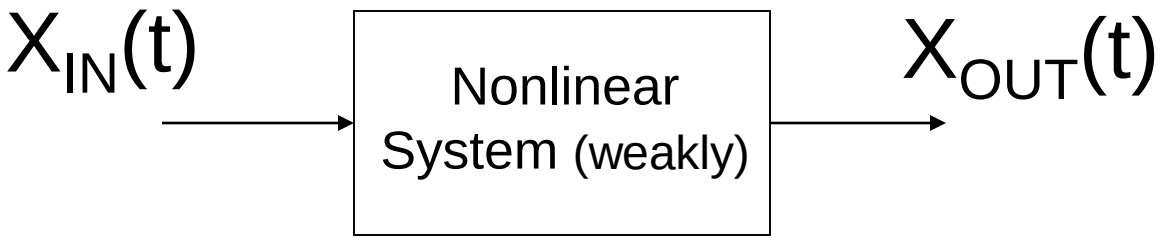
Often the system of interest is ideally linear but practically it is weakly nonlinear.

Often the input is nearly periodic and often sinusoidal and in latter case desired output is also sinusoidal

Weak nonlinearity will cause harmonic distortion (often just termed distortion) of signal as it is propagated through the system

Spectral analysis often used to characterize effects of the weak nonlinearity

Spectral Analysis



Distortion Types:

Frequency Distortion

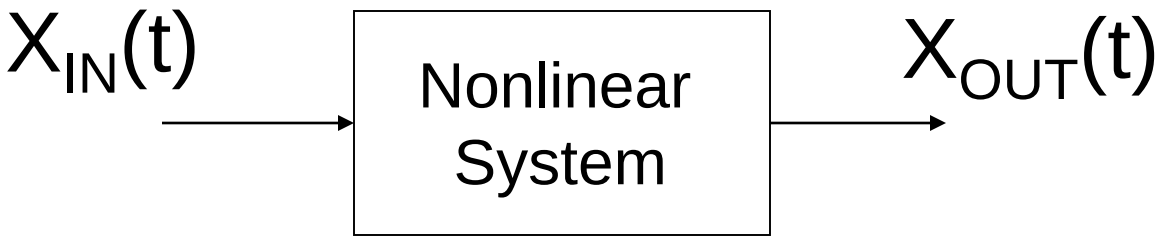
Nonlinear Distortion (alt. harmonic distortion)

Frequency Distortion: Amplitude and phase of system is altered but output is linearly related to input

Nonlinear Distortion: System is not linear, frequency components usually appear in the output that are not present in the input

Spectral Analysis is the characterization of a system with a periodic input with the Fourier series relationships between the input and output waveforms

Spectral Analysis



If $$X_{IN}(t) = X_m \sin(\omega t + \theta)$$

$$X_{OUT}(t) = A_0 + \sum_{k=1}^{\infty} A_k \sin(k\,\omega\,t + \theta_k)$$

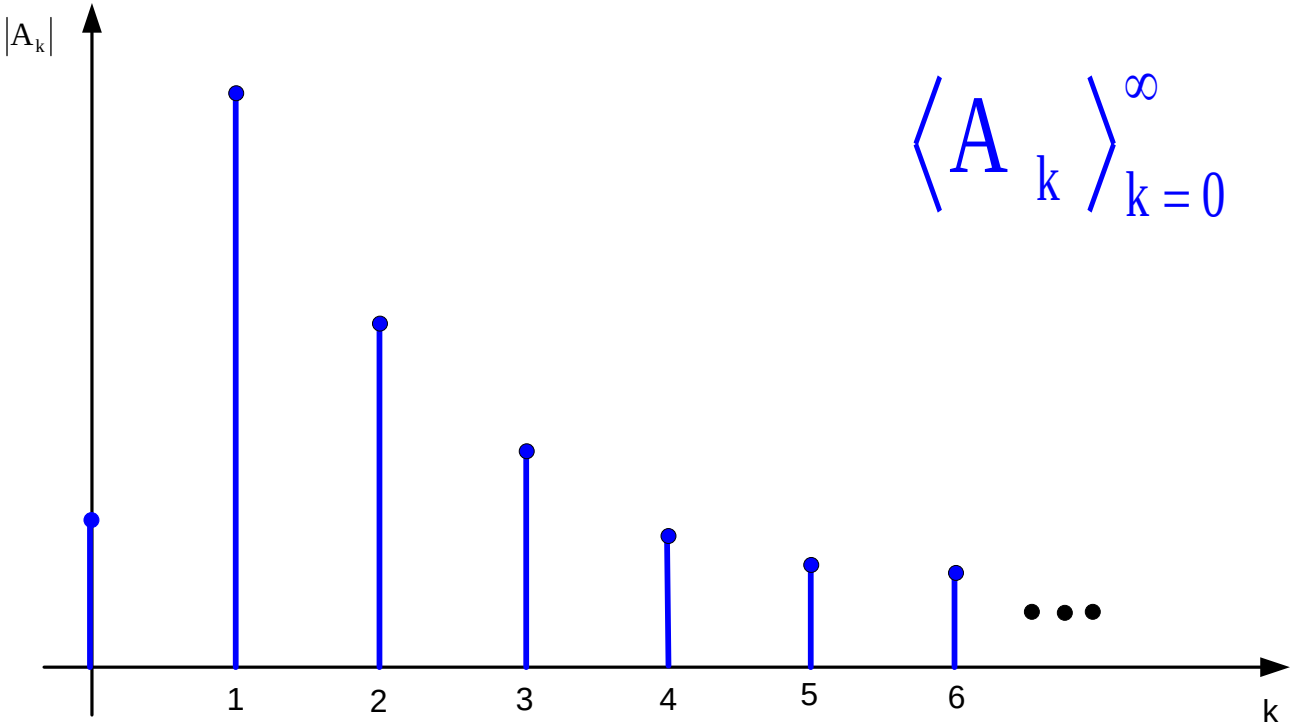
All spectral performance metrics depend upon the sequences $\langle A_k \rangle_{k=0}^{\infty}$ $\langle \theta_k \rangle_{k=1}^{\infty}$
(index sequence, not time sequence)

Typical spectral performance metrics of interest: SNDR, SDR, THD, SFDR, IMOD

Alternately

$$X_{OUT}(t) = A_0 + \sum_{k=1}^{\infty} a_k \sin(k\omega t) + \sum_{k=1}^{\infty} b_k \cos(k\omega t) \qquad A_k = \sqrt{a_k^2 + b_k^2} \qquad \theta_k = \tan^{-1}\left(\frac{b_k}{a_k}\right)$$

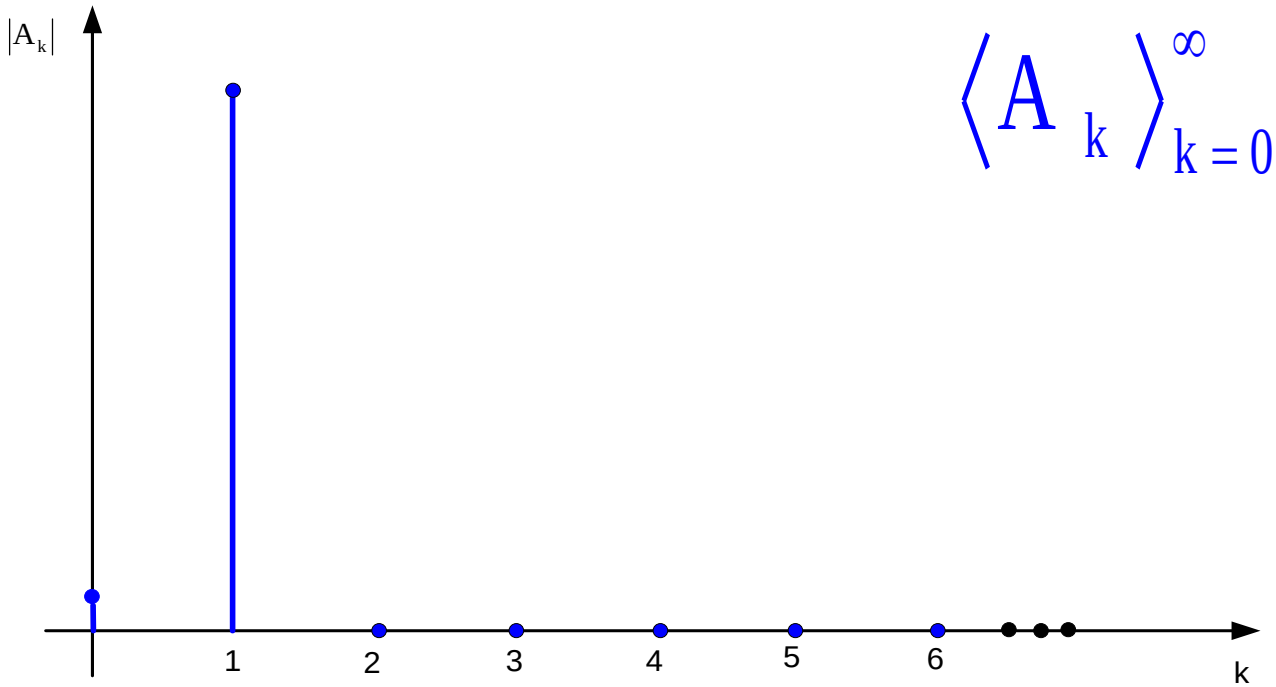
Distortion Analysis



- Often termed the DFT coefficients (will show later)
- Spectral lines, not a continuous function

A_1 is termed the fundamental (when input is sinusoid or periodic)
A_k is termed the kth harmonic (when input is sinusoid or periodic)

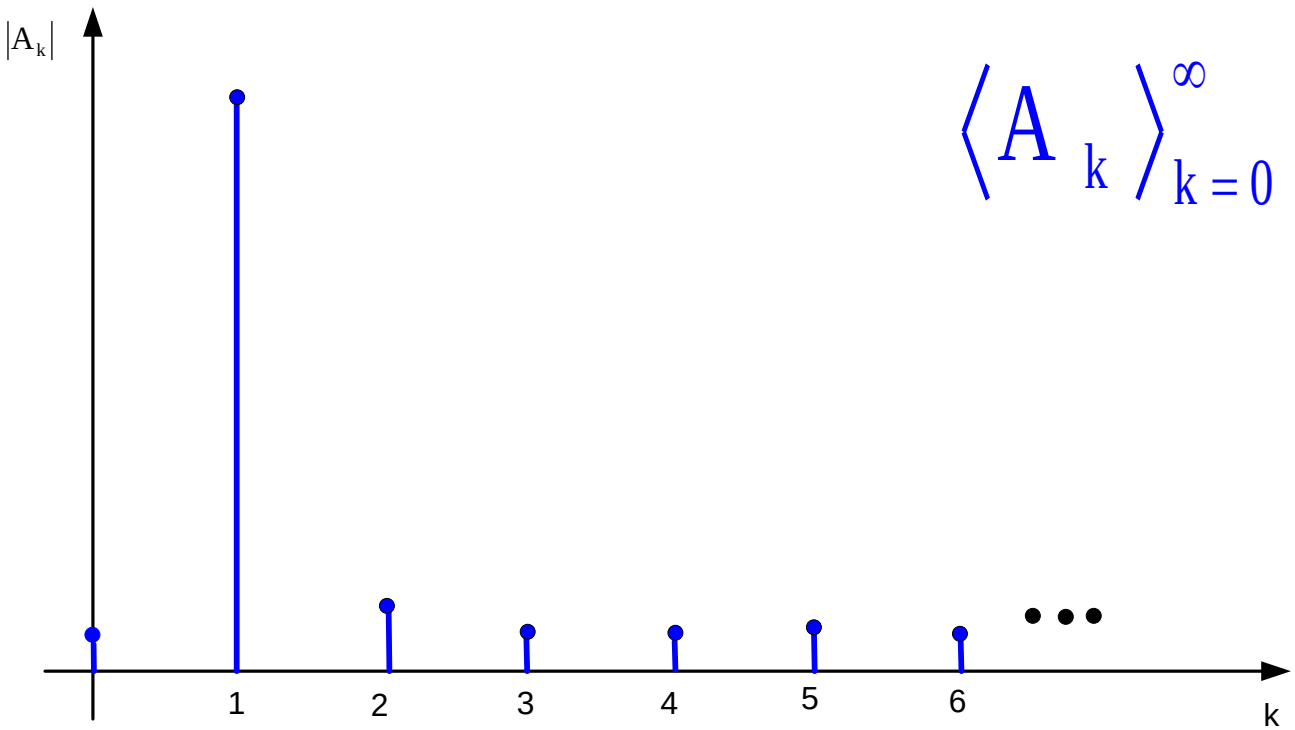
Distortion Analysis



$$\langle A_k \rangle_{k=0}^{\infty}$$

Often <u>ideal</u> response will have only fundamental present and all remaining spectral terms will vanish

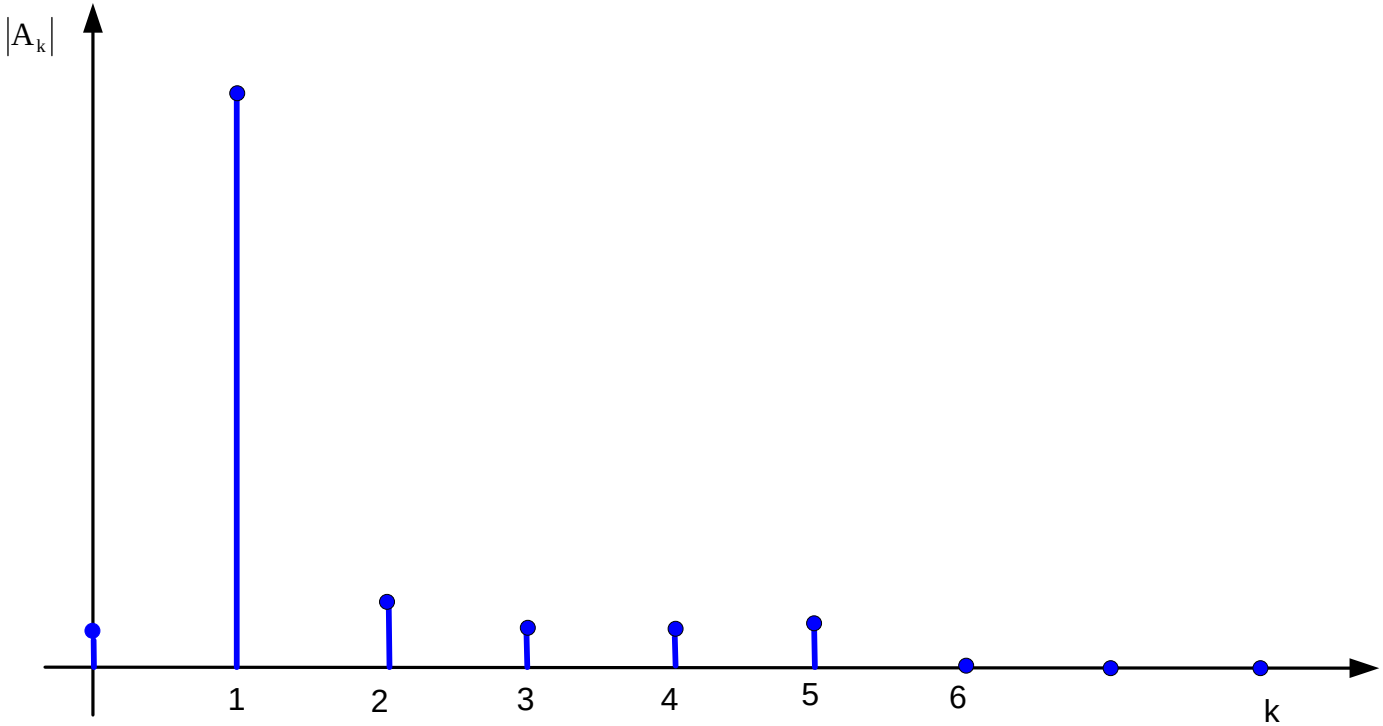
Distortion Analysis



For a low distortion signal, the 2nd and higher harmonics are generally much smaller than the fundamental

The magnitude of the harmonics generally decrease rapidly with k for low distortion signals

Distortion Analysis



Assume f(t) is periodic with period $T = \frac{1}{f}$

f(t) is band-limited to frequency $2\pi f k_X$ if $A_k=0$ for all $k>k_x$

where $\langle A_k \rangle_{k=0}^{\infty}$ are the Fourier series coefficients of f(t)

Distortion Analysis

Total Harmonic Distortion, THD

$$\mathrm{THD} = \frac{\text{RMS voltage in harmonics}}{\text{RMS voltage of fundamental}}$$

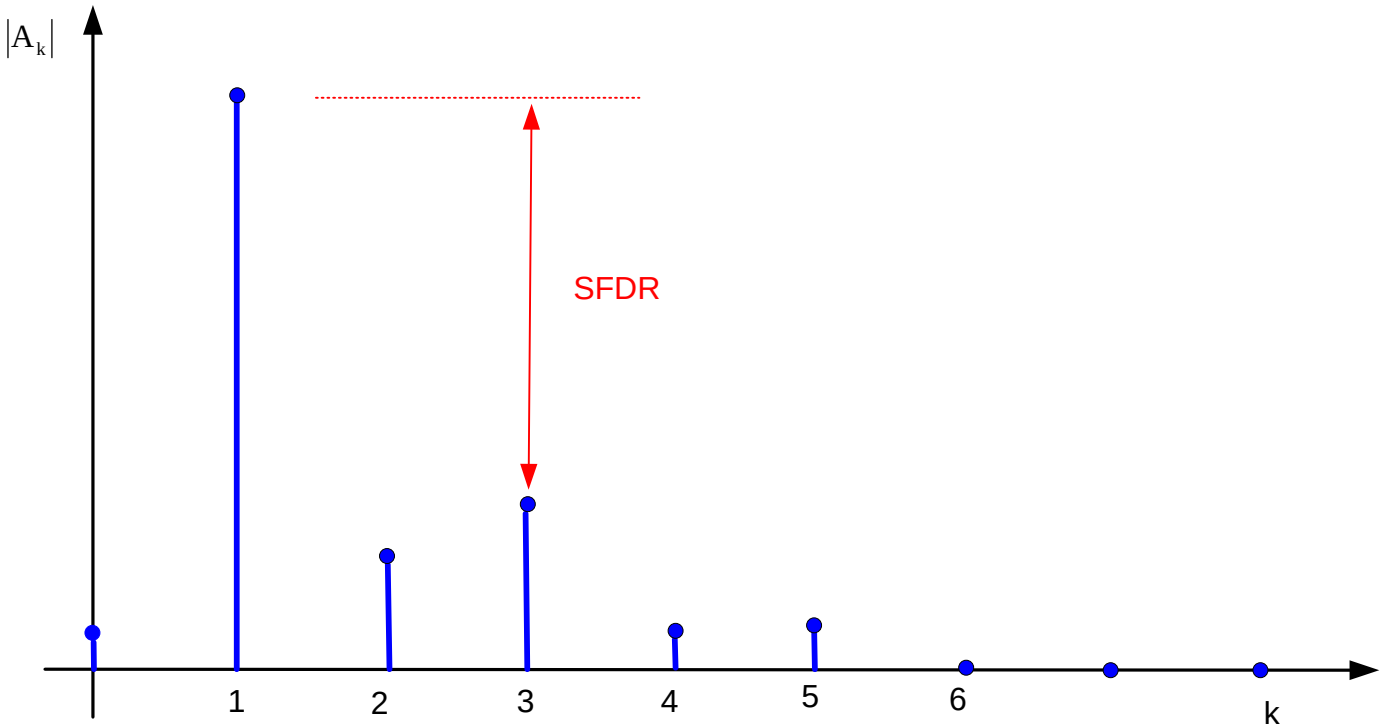
$$\mathrm{THD} = \frac{\sqrt{\left(\frac{A_2}{\sqrt{2}}\right)^2 + \left(\frac{A_3}{\sqrt{2}}\right)^2 + \left(\frac{A_4}{\sqrt{2}}\right)^2 + \ldots}}{\frac{A_1}{\sqrt{2}}}$$

$$\mathrm{THD} = \frac{\sqrt{\sum_{k=2}^{\infty} A_k^2}}{A_1}$$

Distortion Analysis

Spurious Free Dynamic Range, SFDR

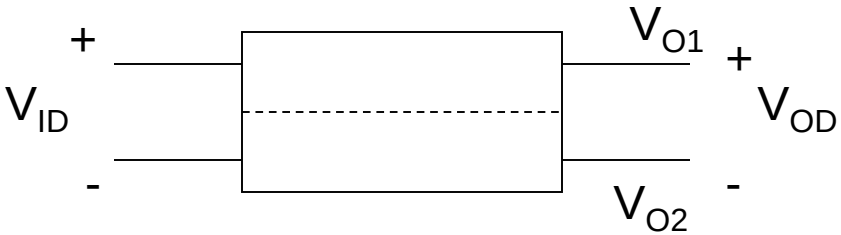
The SFDR is the difference between the fundamental and the largest harmonic



SFDR and THD are usually determined by either the second or third harmonic

Distortion Analysis

Theorem: In a fully differential symmetric circuit, all even-order terms are absent in the Taylor's series output for symmetric differential sinusoidal excitations !



Proof: Expanding in a Taylor's series around V_{ID}=0, we obtain

$$V_{01} = f\left(V_{ID}\right) = \sum_{k=0}^{\infty} h_k V_{ID}^k$$

$$V_{02} = f\left(-V_{ID}\right) = \sum_{k=0}^{\infty} h_k \left(-V_{ID}\right)^k$$

$$V_{OD} = V_{01} - V_{02} = \sum_{k=0}^{\infty} h_k \left(V_{ID}\right)^k - \sum_{k=0}^{\infty} h_k \left(-V_{ID}\right)^k$$

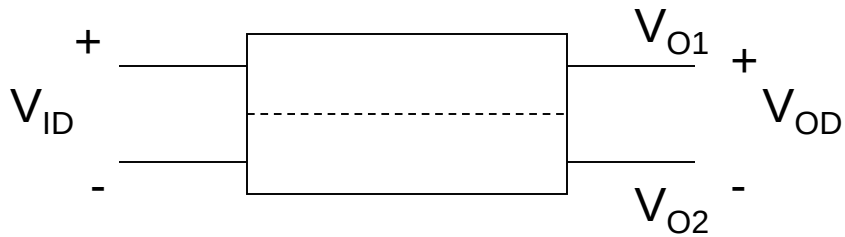
$$V_{OD} = \sum_{k=0}^{\infty} h_k \left[\left(V_{ID}\right)^k - \left(-V_{ID}\right)^k\right]$$

$$V_{OD} = \sum_{k=0}^{\infty} h_k \left[\left(V_{ID}\right)^k - \left(-1\right)^k \left(V_{ID}\right)^k\right]$$

When k is even, the corresponding term in [ ] vanishes

Distortion Analysis

Theorem: In a fully differential symmetric circuit, all even harmonics are absent in the differential output for symmetric differential sinusoidal excitations !



Proof: From prev theorem for $V_{ID} = \alpha \sin(\omega t)$ $\quad V_{OD} = \sum_{\substack{k=0 \\ k \text{ odd}}}^{\infty} \beta_k \sin^k(\omega t)$

Recall:

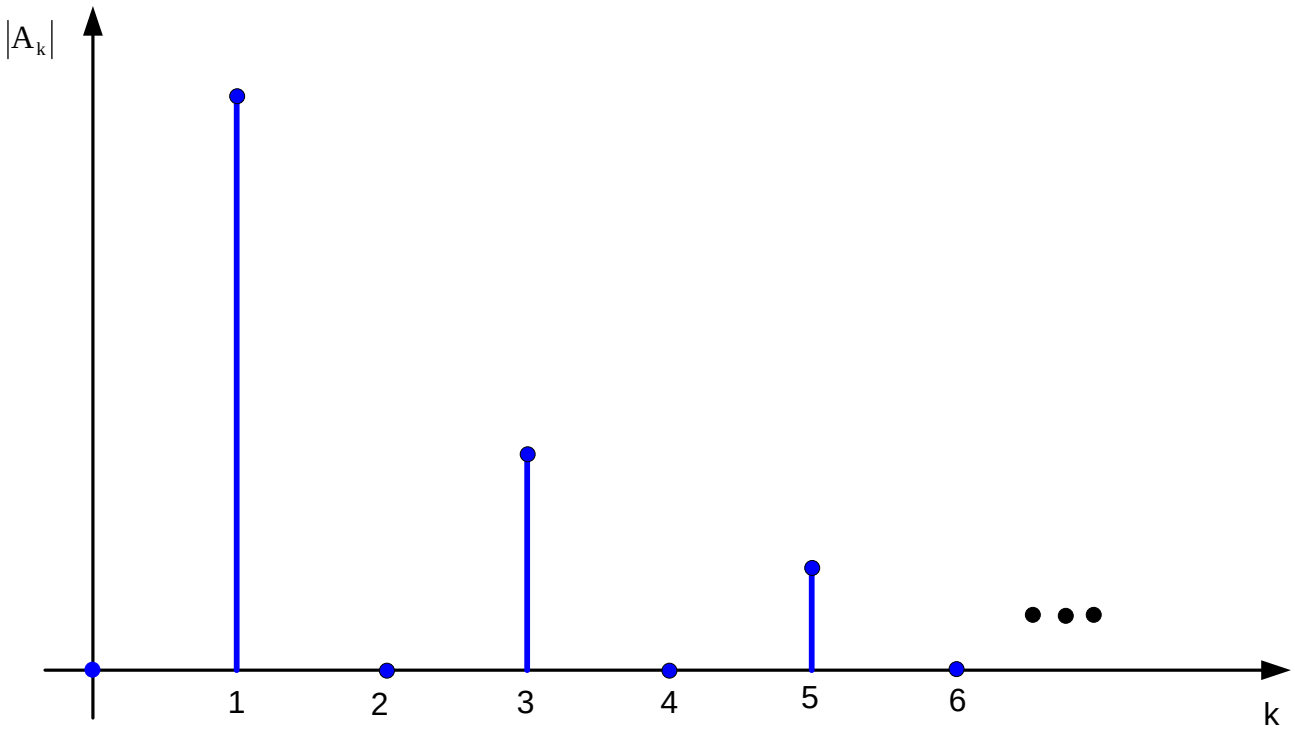
$$\sin^n(x) = \begin{cases} \sum_{k=0}^{\frac{n-1}{2}} h_k \sin\left((n-2k)x\right) & \text{for } n \text{ odd} \\ \sum_{k=0}^{\frac{n-2}{2}} g_k \sin\left((n-2k)x + \theta_k\right) & \text{for } n \text{ even} \end{cases}$$

where h_k, g_k, and θ_k are constants

That is, odd powers of $\sin^n(x)$ have only odd harmonics present and even powers have only even harmonics present

Distortion Analysis

In a fully differential symmetric circuit, all even harmonics are absent in the differential output !



Distortion Analysis

How are spectral magnitude components determined?

By integral

$$A_k = \frac{1}{\omega T}\left(\int_{t_1}^{t_1+T} f(t)e^{-jk\omega t}dt + \int_{t_1}^{t_1+T} f(t)e^{jk\omega t}dt\right)$$

or

$$a_k = \frac{2}{\omega T}\int_{t_1}^{t_1+T} f(t)\sin(kt\omega)dt \qquad b_k = \frac{2}{\omega T}\int_{t_1}^{t_1+T} f(t)\cos(kt\omega)dt$$

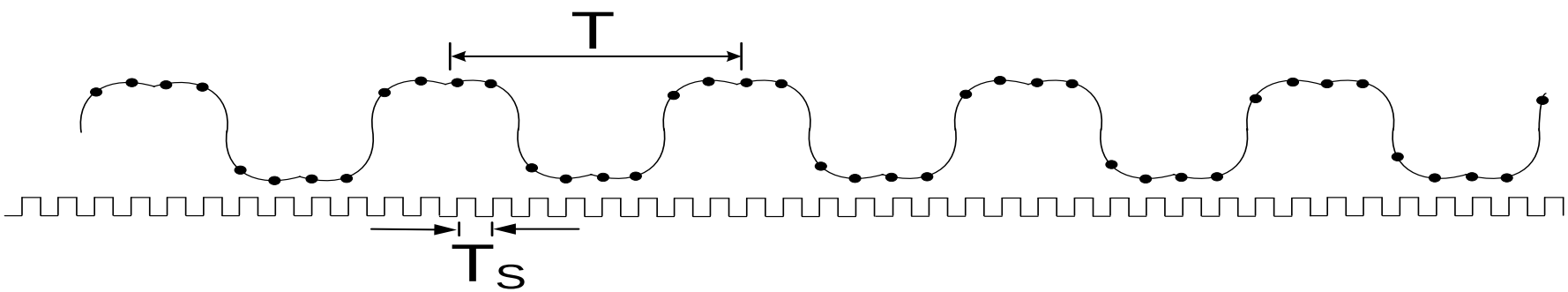
Integral is very time consuming, particularly if large number of components are required

By DFT (with some restrictions that will be discussed)

By FFT (special computational method for obtaining DFT)

Distortion Analysis

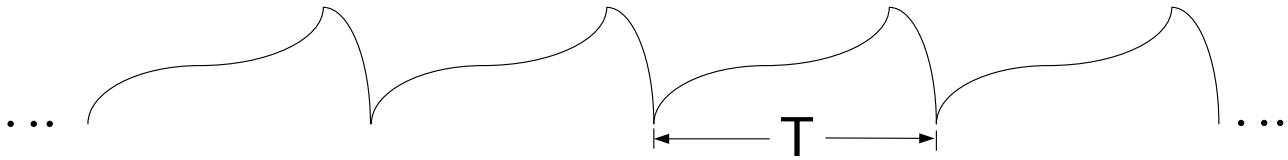
How are spectral components determined?



Consider sampling f(t) at uniformly spaced points in time T_S seconds apart

This gives a sequence of samples $\left\langle f\left(kT_s\right)\right\rangle_{k=1}^{N}$

Distortion Analysis



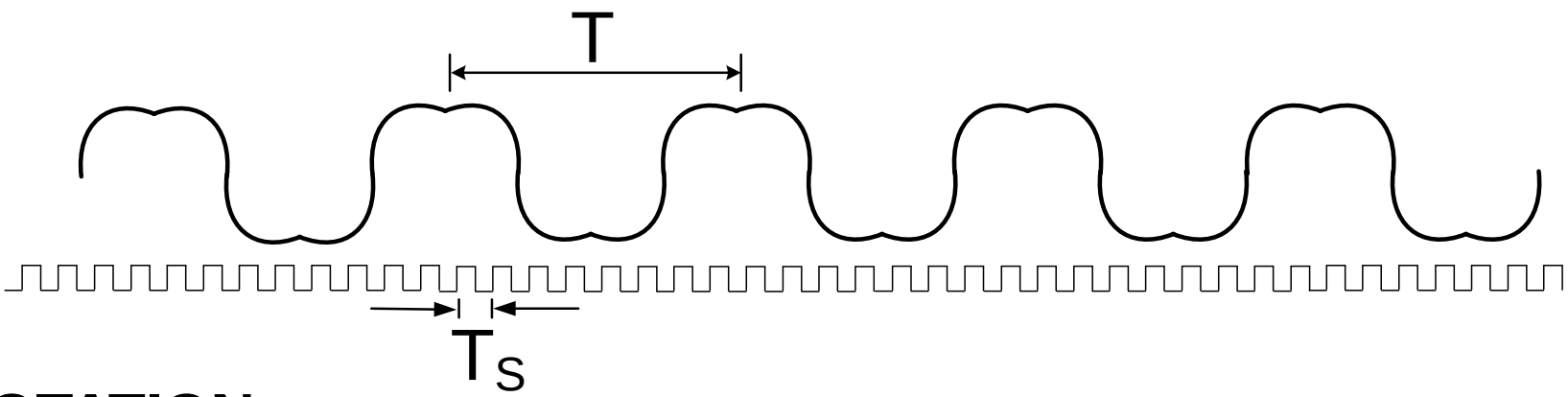
Consider a function f(t) that is periodic with period T

$$f(t) = A_0 + \sum_{k=1}^{\infty} A_k \sin\left(k\omega t + \theta_k\right) \qquad \omega = 2\pi f = \frac{2\pi}{T}$$

Band-limited Periodic Functions

Definition: A periodic function of frequency f is band limited to a frequency f_{max} if $A_k=0$ for all $k > \frac{f_{max}}{f}$

Distortion Analysis



NOTATION:

T: Period of Excitation

T_S: Sampling Period

N_P: Number of periods over which samples are taken

N: Total number of samples

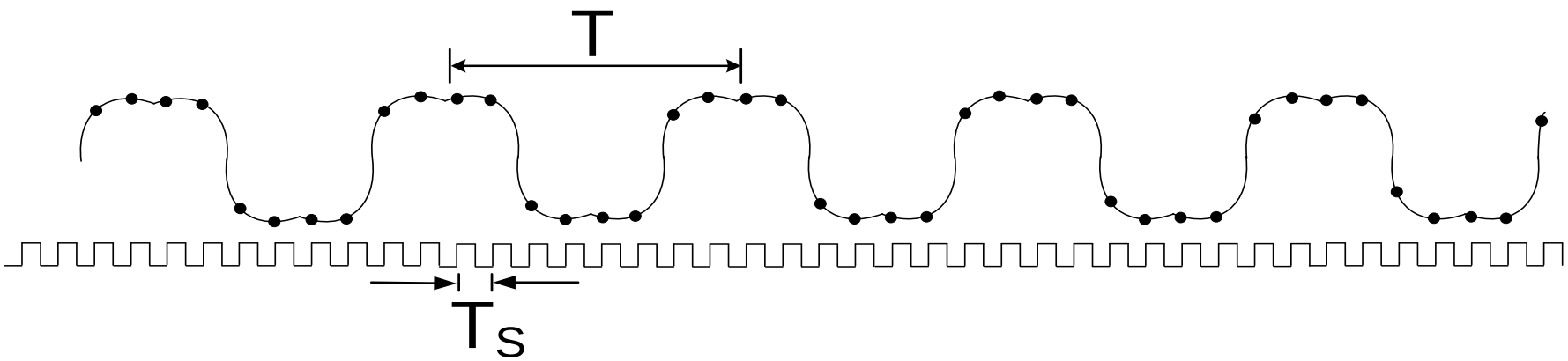
$$N_P = \frac{NT_S}{T}$$

Note: N_P is not an integer unless a specific relationship exists between N, T_S and T

$$h = \mathrm{Int}\left(\left[\frac{N}{2}-1\right]\frac{1}{N_P}\right)$$

Note: The function Int(x) is the integer part of x

Distortion Analysis



THEOREM: If N_P is an integer and x(t) is band limited to f_{MAX}, then

$$|A_m| = \frac{2}{N}\left|X(mN_P + 1)\right| \qquad 0 \le m \le h-1$$

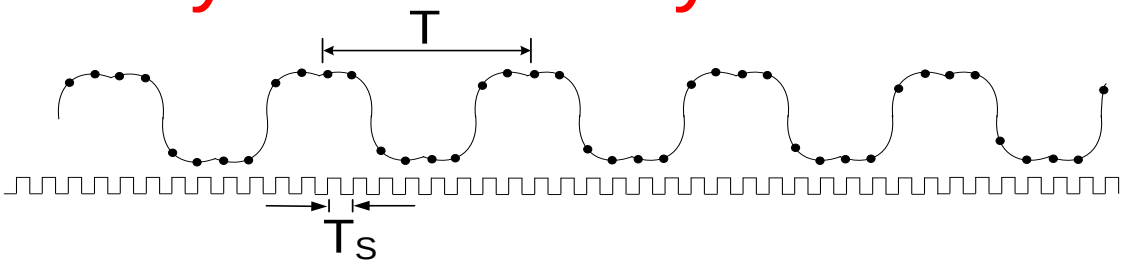
and $X(k) = 0$ **for all k not defined above**

where $\langle X(k)\rangle_{k=0}^{N-1}$ **is the DFT of the sequence** $\langle x(kT_S)\rangle_{k=0}^{N-1}$

f = 1/T, $f_{MAX} = \frac{f}{2}\bullet\left[\frac{N}{N_P}\right]$ **, and** $h = \mathrm{Int}\left(\left[\frac{N}{2}-1\right]\frac{1}{N_P}\right)$

Key Theorem central to Spectral Analysis that is widely used !!! and often “abused”

Why is this a Key Theorem?



THEOREM: If N_P is an integer and x(t) is band limited to f_{MAX}, then

$$|A_m| = \frac{2}{N}|X(mN_P + 1)| \qquad 0 \le m \le h-1$$

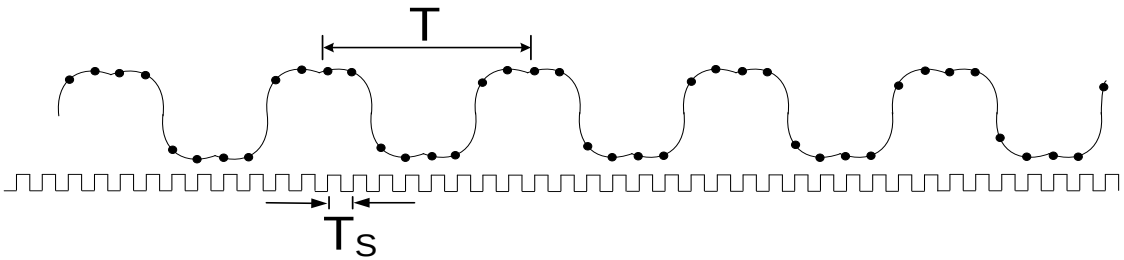
and $X(k) = 0$ **for all k not defined above**

where $\langle X(k)\rangle_{k=0}^{N-1}$ **is the DFT of the sequence** $\langle x(kT_S)\rangle_{k=0}^{N-1}$

f = 1/T, $f_{MAX} = \frac{f}{2} \bullet \left[\frac{N}{N_P}\right]$ **, and** $h = \mathrm{Int}\left(\left[\frac{N}{2} - 1\right]\frac{1}{N_P}\right)$

- DFT requires dramatically less computation time than the integrals for obtaining Fourier Series coefficients
- Can easily determine the sampling rate (often termed the Nyquist rate) to satisfy the band limited part of the theorem ($f_S > 2f_{MAX}$)

How is this theorem abused?



THEOREM: If N_P is an integer and x(t) is band limited to f_{MAX}, then

$$|A_m| = \frac{2}{N}|X(mN_P + 1)| \qquad 0 \le m \le h-1$$

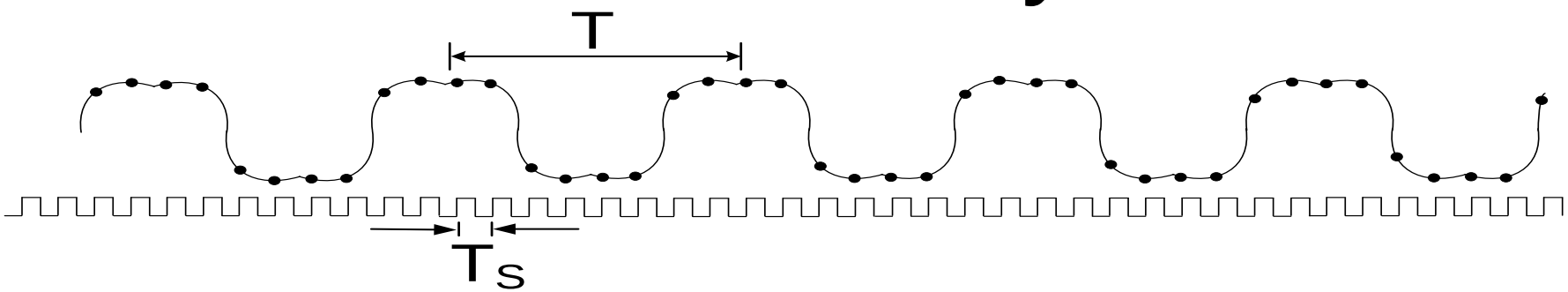
and $X(k) = 0$ **for all k not defined above**

where $\langle X(k)\rangle_{k=0}^{N-1}$ **is the DFT of the sequence** $\langle x(kT_S)\rangle_{k=0}^{N-1}$

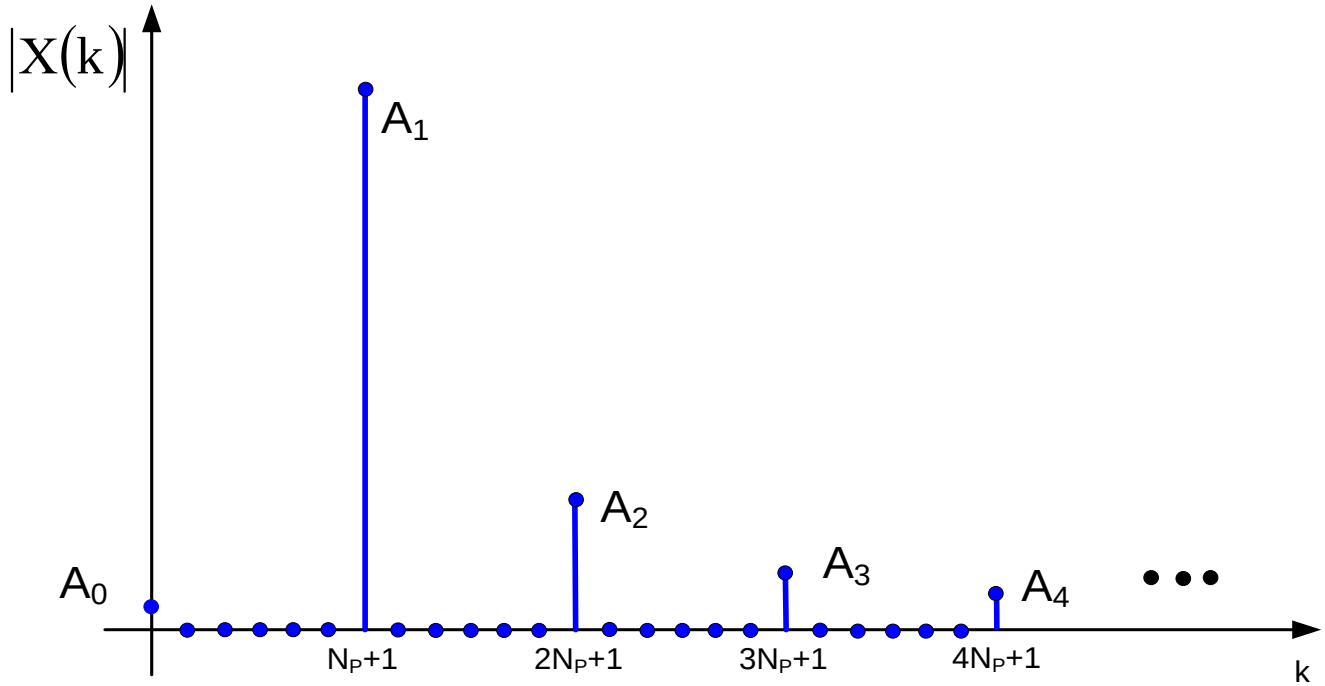
f = 1/T, $f_{MAX} = \frac{f}{2} \bullet \left[\frac{N}{N_P}\right]$ **, and** $h = \mathrm{Int}\left(\left[\frac{N}{2} - 1\right]\frac{1}{N_P}\right)$

- Much evidence of engineers attempting to use the theorem when N_P is not an integer
- Challenging to have N_P an integer in practical applications
- Dramatic errors can result if there are not exactly an integer number of periods in the sampling window

Distortion Analysis

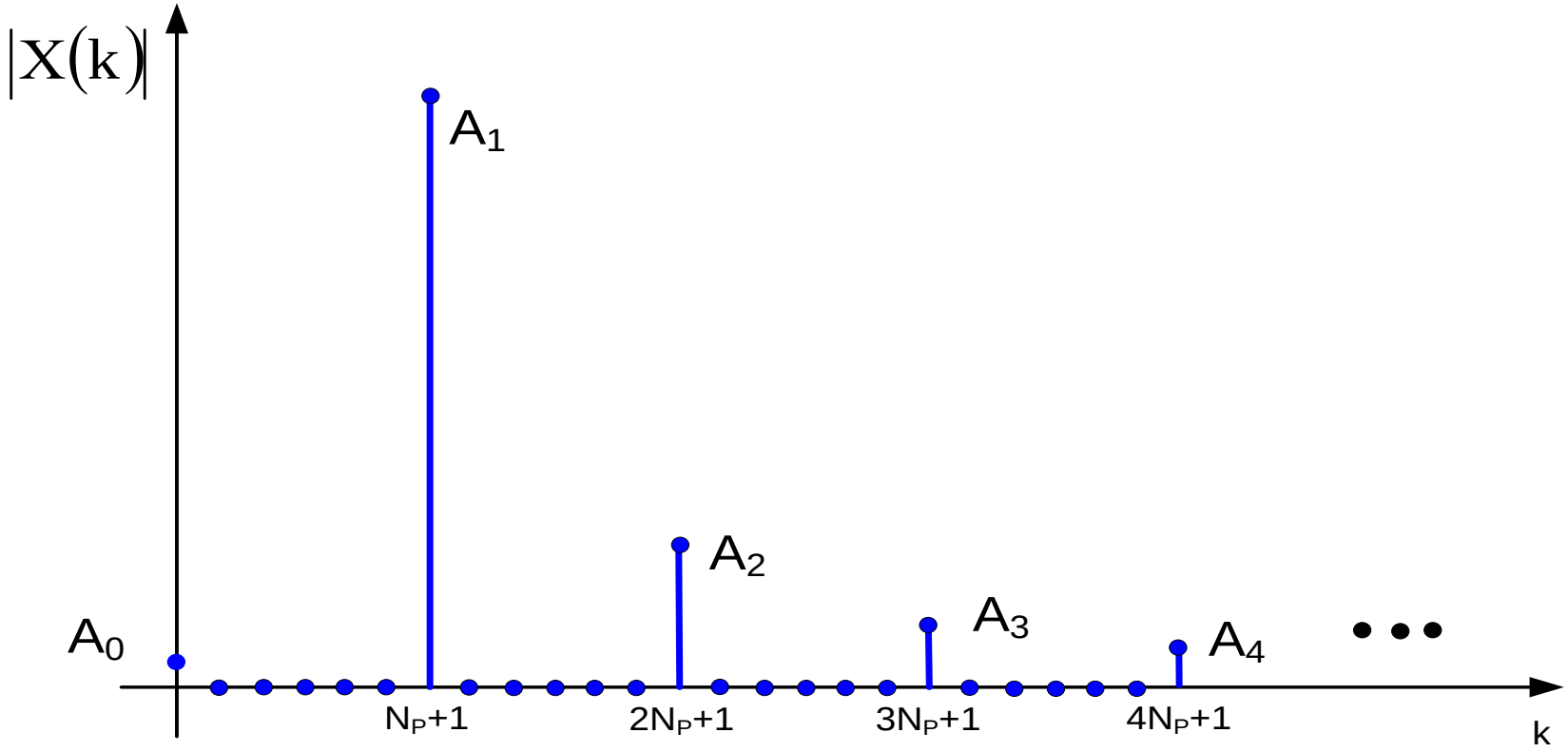


If the hypothesis of the theorem are satisfied, we thus have



Distortion Analysis

If the hypothesis of the theorem are satisfied, we thus have



FFT is a computationally efficient way of calculating the DFT, particularly when N is a power of 2

Stay Safe and Stay Healthy !

End of Lecture 4